Markscheme

November 2020

Discrete mathematics

Higher level

Paper 3

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.

- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

  Examples

	Correct answer seen	**Further working seen**	**Action**
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working*.

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets eg (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Misread

*If a candidate incorrectly copies information from the question, this is a misread (**MR**).*
*A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses **[1 mark]**.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives

$$f'(x) = \left(2\cos(5x-3)\right)5 \left(= 10\cos(5x-3)\right) \qquad \textbf{\textit{A1}}$$

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

Candidates should ***NO LONGER*** *be penalized for an accuracy error* ***(AP)***.

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for ***FT***.

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required for paper 2, but calculators with symbolic manipulation features (for example, TI-89) are not allowed.

Calculator notation
The Mathematics HL guide says:

Students must always use correct mathematical notation, not calculator notation.

Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

14. Candidate work

Candidates are meant to write their answers to Section A on the question paper (QP), and Section B on answer booklets. Sometimes, they need more room for Section A, and use the booklet (and often comment to this effect on the QP), or write outside the box. This work should be marked.

The instructions tell candidates not to write on Section B of the QP. Thus they may well have done some rough work here which they assume will be ignored. If they have solutions on the answer booklets, there is no need to look at the QP. However, if there are whole questions or whole part solutions missing on answer booklets, please check to make sure that they are not on the QP, and if they are, mark those whole questions or whole part solutions that have not been written on answer booklets.

1. the auxiliary equation is $\lambda^2 - 8\lambda + 16 = 0$ (or equivalent) ***A1***

attempts to solve their quadratic equation ***(M1)***

$\lambda = 4$ ***A1***

the general solution is of the form $u_n = (A + Bn)4^n$ (or equivalent) ***M1***

Note: Only award ***M1*** for general solutions of the form $u_n = (A + Bn)\lambda^n$ (or equivalent). Award ***M1*** for $u_n = (A + Bn)\lambda^n$.

applies two (initial) conditions, eg substitutes $n = 1, 2$, into their u_n to form two equations ***M1***

$A + B = 1$ and $A + 2B = 3$ (or equivalent *eg* $4A + 4B = 4$ and $16A + 32B = 48$) ***A1***

attempts to solve their two simultaneous equations ***(M1)***

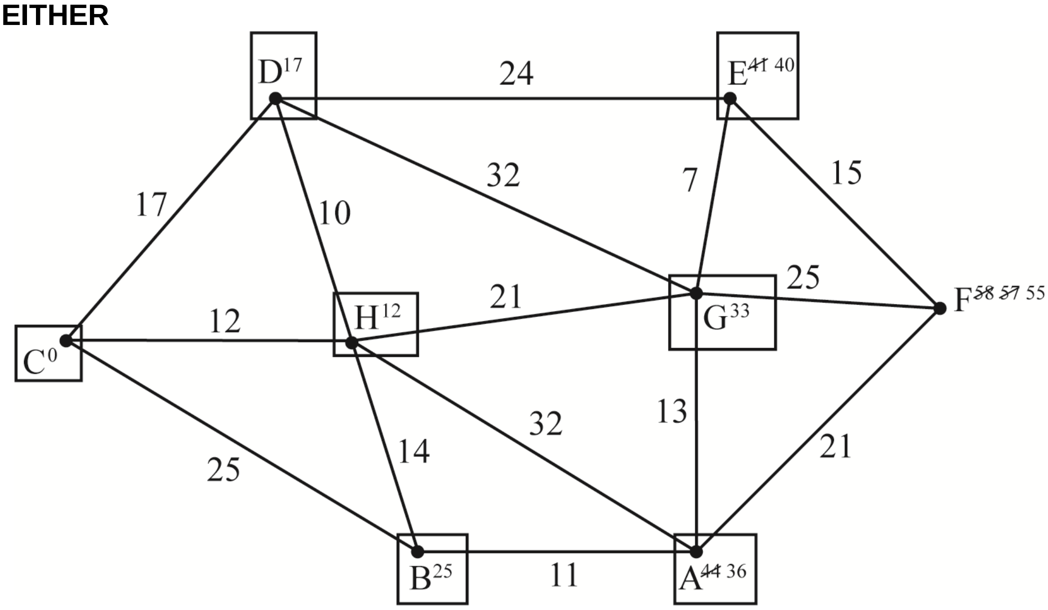
$A = -1, B = 2$

$u_n = (2n - 1)4^n$ (or equivalent) ***A1***

[8 marks]

2. (a) (i) attempts to construct a graph or table to represent Dijkstra's algorithm ***M1***

EITHER



OR

Step	**A**	**B**	**C**	**D**	**E**	**F**	**G**	**H**
1	∞	25	0	17	∞	∞	∞	12
2	44	25	-	17	∞	∞	33	-
3	44	25	-	-	41	∞	33	-
4	36	-	-	-	41	∞	33	-
5	36	-	-	-	40	58	-	-
6	-	-	-	-	40	57	-	-
7	-	-	-	-	-	55	-	-

a clear attempt at Step 1 ($\mathrm{C,D,H}$ and B considered) ***M1***

Steps 2 and 3 correctly completed ***A1***

Step 4 (A: $44 \to 36$) correctly completed ***A1***

Steps 5 and 6 (E: $41 \to 40$ and F: $58 \to 57$ or $57 \to 55$ or $58 \to 55$) correctly completed ***A1***

shortest time $= 55$ (mins) ***A1***

(ii) CHGEF ***A1***

Note: Award ***A1*** only if it is clear that Dijkstra's algorithm has been attempted in part (a) (i). This ***A1*** can be awarded if the candidate attempts to use Dijkstra's algorithm but neglects to state 55 (mins).

[7 marks]

(b) minimum travel time from C to A is reduced

CHA is now $12+t$ (mins) ***(M1)***

CBA is still $25+11$ (mins)

so $12+t<36$ $(t<24)$ ***(A1)***

Note: Condone $t \leq 24$.

travel time from C to F remains the same (55 mins)

CHAF is now $12+t+21$ (mins) ***(M1)***

$12+t+21 \geq 55$ $(t \geq 22)$ ***(A1)***

so $22 \leq t < 24$ ***A1***

Note: Accept $t = 22, 23$.

[5 marks]
Total [12 marks]

3. (a) (i) the remainder is 0 ***A1***

(ii) $14^{16} \equiv 1(\text{mod}\,17)$ (from Fermat's little theorem) ***(A1)***

$14^{2022} = 14^{16\times126+6}$ ***(M1)***

Note:	Award ***M1*** for a RHS exponent consistent with the correct use of Fermat's little theorem.

$14^{2022} \equiv 14^{6}(\text{mod}\,17)$ $\left(\equiv 15(\text{mod}\,17)\right)$ ***A1***

the remainder is 15 ***A1***

[5 marks]

(b) (i) **METHOD 1**

let $N = a_n 13^n + a_{n-1}13^{n-1} + ... + a_1 13 + a_0$ ***M1***

Note:	The above ***M1*** is independent of the ***A*** marks below.

$13 \equiv 1(\text{mod}\,6)$ ***A1***

EITHER

$13^x \equiv 1(\text{mod}\,6)$ (for all $x \in \mathbb{N}$) ***A1***

OR

$N \equiv a_n 1^n + a_{n-1}1^{n-1} + ... + a_1 1 + a_0(\text{mod}\,6)$ $\left(N \equiv a_n + a_{n-1} + ... + a_1 + a_0(\text{mod}\,6)\right)$ ***A1***

THEN

so $N \equiv 0(\text{mod}\,6)$ if and only if $a_n + a_{n-1} + ... + a_1 + a_0 \equiv 0(\text{mod}\,6)$ ***R1***

so $6 \mid N$ if and only if $6 \mid (a_n + a_{n-1} + ... + a_1 + a_0)$ ***AG***

METHOD 2

let $N = a_n 13^n + a_{n-1} 13^{n-1} + \ldots + a_1 13 + a_0$ ***(M1)***

$N = (a_n + a_{n-1} + \ldots + a_1 + a_0)$
$+(13-1)\left(a_n\left(13^{n-1} + \ldots + 13^0\right) + a_{n-1}\left(13^{n-2} + \ldots + 13^0\right) + \ldots + a_1 13^0\right)$ ***M1A1***

Note: Award ***M1*** for attempting to express N in the form $N = (a_n + a_{n-1} + \ldots + a_1 + a_0) + (13-1)M$.	

as $6 \mid (13-1)M$ ***R1***

so $6 \mid N$ if and only if $6 \mid (a_n + a_{n-1} + \ldots + a_1 + a_0)$ ***AG***

(ii) **METHOD 1**

the sum of the digits is $2y + 20$ ***(A1)***

uses $2y + 20 = 6k$ (or equivalent) to attempt to find a valid value of y ***(M1)***

$y = 2, 5, 8, 11(B)$ ***A1A1***

Note: Award ***A1*** for $y = 2, 5, 8$ and ***A1*** for $y = 11(B)$.	

METHOD 2

$(1y93y25)_{13} = 1\times13^6 + y\times13^5 + 9\times13^4 + 3\times13^3 + y\times13^2 + 2\times13^1 + 5\times13^0$ ***(A1)***

$= 371462y + 5090480$

attempts to find a valid value of y such that
$371462y + 5090480 \equiv 0 \pmod 6$ ***(M1)***

$y = 2,5,8,11(B)$ ***A1A1***

Note: Award ***A1*** for $y = 2,5,8$ and ***A1*** for $y = 11(B)$.

[8 marks]
Total [13 marks]

4. METHOD 1

$3x \equiv 1 \pmod{31} \Rightarrow x \equiv 21 \pmod{31}$ ***(A1)***

$x = 29a + 7$ and $x = 31b + 21$ ***(M1)***

uses a table of values for their two equations in an attempt to find x ***(M1)***

$x = 703$ ***(A2)***

$x \equiv 703 \pmod{899}$ $(x = 703 + 899k)$ ***A1***

METHOD 2

$x = 29a + 7$ ***(A1)***

substitutes their equation into the other congruence ***(M1)***

$3(29a + 7) \equiv 1 \pmod{31}$

a valid attempt to solve their form of the above congruence ***(M1)***

$a \equiv 24 \pmod{31}$ ***(A1)***

correctly substitutes $a = 31b + 24$ into $x = 29a + 7$ ***(A1)***

$x = 29(31b + 24) + 7$

$x \equiv 703 \pmod{899}$ $(x = 703 + 899b)$ ***A1***

METHOD 3

$3x \equiv 1 \pmod{31} \Rightarrow x \equiv 21 \pmod{31}$ ***(A1)***

$M_1 = 31$ and $M_2 = 29$ ***(A1)***

attempts to use $M_i x_i \equiv 1 \pmod{m_i}$ $(i = 1, 2)$ ***(M1)***
$31x_1 \equiv 1 \pmod{29}$ and $29x_2 \equiv 1 \pmod{31}$

$x_1 \equiv 15 \pmod{29}$ and $x_2 \equiv 15 \pmod{31}$ ***(A1)***

attempts to use $x = a_1 x_1 M_1 + a_2 x_2 M_2 \pmod{M}$ ***(M1)***

$x \equiv 7 \times 15 \times 31 + 21 \times 29 \times 15 \pmod{899}$

$x \equiv 703 \pmod{899}$ $\left(x \equiv 12390 \pmod{899}\right)$ $(x = 703 + 899k)$ ***A1***

METHOD 4

$x = 29k + 7$ and $3x = 31t + 1$ ***(M1)***

p is a multiple of 31 and $p = 29k + 7$ (1) and

q is a multiple of 29 and $3q = 31t + 1$ (2) ***(M1)***

from (1): $p = 31x_1$ and $31x_1 \equiv 7 \pmod{29} \Rightarrow x_1 \equiv 18 \pmod{29}$ ***(A1)***

from (2): $q = 29x_2$ and $3(29x_2) \equiv 1 \pmod{31} \Rightarrow x_2 \equiv 5 \pmod{31}$ ***(A1)***

$p = 558$ and $q = 145$

uses $x = p + q$ to obtain $x = 703$ ***(A1)***

$x \equiv 703 \pmod{899}$ $(x = 703 + 899b)$ ***A1***

METHOD 5

$3x \equiv 1 (\bmod 31) \Rightarrow x \equiv 21 (\bmod 31)$ ***(A1)***

$x = 29a + 7$ and $x = 31b + 21$ ***(M1)***

$29a + 7 = 31b + 21 \Rightarrow 29a - 31b = 14$

$\gcd(29, 31) = 1$ and so $29a - 31b = 1$

a valid attempt to solve either $29a - 31b = 14$ or $29a - 31b = 1$
(*eg* uses the Euclidean algorithm and reverses the process) ***(M1)***

$29 \times 210 - 31 \times 196 = 14$ ***(A1)***

$a = 210$ (and $b = 196$) ***(A1)***

$x \equiv 703 (\bmod 899)$ $\left(x \equiv 6097 (\bmod 899)\right)$ $\left(x = 703 + 899k\right)$ ***A1***

[6 marks]

5. (a) substitutes $v=9$ into either $e=3v-6$ or $e\le 3v-6$ ***(M1)***
the maximum number of edges is 21 $(e\le 21)$ ***A1***

[2 marks]

(b) κ_9 has $\left(\binom{9}{2}=\right)$ 36 edges ***(A1)***

so $e'=36-e$ $\left(=\binom{9}{2}-e\right)$ ***A1***

[2 marks]

(c) $e'\le 21\Rightarrow 36-e\le 21$ ***(M1)***

$15\le e\le 21$ (the possible values are 15, 16, 17, 18, 19, 20 and 21) ***A1***

[2 marks]

(d) recognises that $e+e'=\frac{v(v-1)}{2}$ (or equivalent) ***(A1)***

uses $e \le 3v-6$ and $e' \le 3v-6$ ***M1***

to form $\frac{v(v-1)}{2}-(3v-6) \le 3v-6$ ***A1***

Note: Award ***A1*** for $\frac{v(v-1)}{2}-(3v-6)=3v-6$.

attempts to solve their quadratic inequality (equality) ***(M1)***

$v^2-13v+24 \le 0 \Rightarrow 2.228... \le v \le 10.77...$

the maximum possible value of v is 10 $(v \le 10)$ ***A1***

[5 marks]
Total [11 marks]